Lecture 5 : Continuous Functions

Definition 1 We say the function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

(i.e. we can make the value of $f(x)$ as close as we like to $f(a)$ by taking x sufficiently close to a).

Example Last day we saw that if $f(x)$ is a polynomial, then f is continuous at a for any real number a since $\lim_{x \to a} f(x) = f(a)$.

If f is defined for all of the points in some interval around a (including a), the definition of continuity means that the graph is continuous in the usual sense of the word, in that we can draw the graph as a continuous line, without lifting our pen from the page.

Note that this definition implies that the function f has the following three properties **if f is continuous at** a:

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$. (Note that this implies that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and are equal).

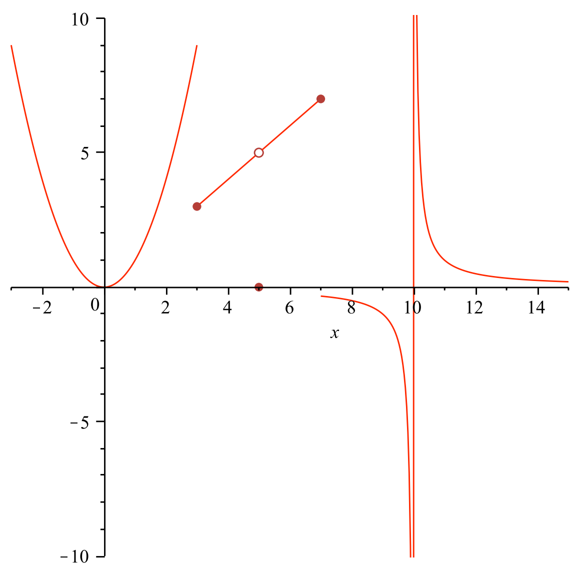
Types of Discontinuities

If a function f is defined near a (f is defined on an open interval containing a, except possibly at a), we say that f is discontinuous at a (or has a discontinuiuty at a) if f is not continuous at a. This can happen in a number of ways. In the graph below, we have a catalogue of discontinuities. Note that a function is discontinuous at a if at least one of the properties 1-3 above breaks down.

Example 2 Consider the graph shown below of the function

$$k(x) = \begin{cases} x^2 & -3 < x < 3 \\ x & 3 \le x < 5 \\ 0 & x = 5 \\ x & 5 < x \le 7 \\ \frac{1}{x-10} & x > 7 \end{cases}$$

Where is the function discontinuous and why?



We say $f(x)$ has a removable discontinuity at a if we can remove the discontinuity at a, by changing the value of the function at a (making $f(a) = \lim_{x\to a} f(x)$). Which of the above discontinuities are removable discontinuities?

Discontinuities where the graph has a vertical asymptote are called infinite discontinuities. Which of the discontinuities above are infinite discontinuities?

Those discontinuities where the graph jumps are called jump discontinuities. Which of the discontinuities above are jump discontinuities?

Definition A function f is continuous from the right at a number a if $\lim_{x\to a^+} = f(a)$.
A function f is continuous from the left at a number a if $\lim_{x\to a^-} = f(a)$.

Example 3 Consider the function $k(x)$ in example 2 above. Ar which of the following x-values is $k(x)$ continuous from the right?

$$x = 0, \quad x = 3, \quad x = 5, \quad x = 7, \quad x = 10.$$

At which of the above x-values is $k(x)$ continuous from the left?

Definition A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right or continuous from the left* at the endpoint as appropriate.)

Example Consider the function $k(x)$ in example 2 above.
(a) On which of the following intervals is $k(x)$ continuous?

$$(-\infty, 0], \quad (-\infty, 3), \quad [3, 7].$$

(b) Fill in the missing endpoints and brackets which give the largest intervals on which $k(x)$ is continuous.

$(-\infty,$ $\qquad$ $(5,$

Example Let

$$m(x) = \begin{cases} cx^2 + 1 & x \geq 2 \\ 10 - x & x < 2 \end{cases}$$

For which value of c is $m(x)$ a continuous function?

Catalogue of functions continuous on their domains

From the last day we know:
Polynomials and Rational functions

- A polynomial function, $P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$, is continuous everywhere i.e. $\lim_{x \to a} P(x) = P(a)$ for all real numbers a.
- A rational function, $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials is continuous on its domain, i.e. $\lim_{x \to a} f(x) = \frac{P(a)}{Q(a)}$ for all values of a, where $Q(a) \neq 0$.

n th Root function
From #10 in last day's lecture, we also have that if $f(x) = \sqrt[n]{x}$, where n is a positive integer, then $f(x)$ is continuous on the interval $[0, \infty)$. We can use symmetry of graphs to extend this to show that $f(x)$ is continuous on the interval $(-\infty, \infty)$, when n is odd. Hence all n th root functions are continuous on their domains.

Trigonometric Functions
In the appendix we provide a proof of the following Theorem :

Theorem 1 The functions $\sin x$ and $\cos x$ are continuous on the interval $(-\infty, \infty)$. In particular; for any real number a, we can evaluate the limits below by direct substitution

$$\lim_{x \to a} \sin x = \sin a, \qquad \lim_{x \to a} \cos x = \cos a.$$

Combining this with Theorem 2 below will show that all of the trigonometric function $\sin x, \cos x$, $\tan x, \sec x$, $\csc x, \cot x$ are continuous on their domains.

From Theorem 2 below we get that **functions which are algebraic combinations of the functions using $+, -, \cdot$ and $\div$ listed above are also continuous on their domains.**

Theorem 2 If f and g are continuous at a and c is constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf **4.** fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$.

It is relative easy to prove this theorem using the limit laws from the previous lecture. A sample proof of number 5 is given in the appendix.

> **Note** Collecting the above results, we can show that the following types of functions and combinations of them using $+, -, \cdot$ and $\div$ are continuous on their domains:
>
> Polynomial Functions   Rational functions
> Root Functions   Trigonometric functions

Example Find the domain of the following function and use the theorem above to show that it is continuous on its domain: $g(x) = \frac{(x^2+3)^2}{x-10}$.

Example Find the domain of the following function and use the theorem above to show that it is continuous on its domain:

$$k(x) = \sqrt[3]{x}(x^2 + 2x + 1) + \frac{x+1}{x-10}.$$

$k(x)$ is continuous on its domain, since it is a combination of root functions, polynomials and rational functions using the operations $+, -, \cdot$ and $\div$. The domain of k is all values of x except $x = 10$ and this function is continuous on the intervals $(-\infty, 10)$ and $(10, \infty)$.

Example Use Theorem 1 and property 5 from theorem 2 above to show that $\tan x$ is continuous on its domain.

$\tan(x)$ is continuous on its domain, since it is a combination of the functions $\sin x$ and $\cos x$ (both of which are continuous for all x) using the operation $\div$. The domain of $\tan x$ is all values of x except those where $\cos x = 0$, that is all values of x except the odd multiples of $\frac{\pi}{2}$.

Example: Removable Discontinuity Recall that last day we found $\lim_{x\to 0} x^2 \sin(1/x)$ using the squeeze theorem. What is the limit?

Does the function

$$n(x) = \begin{cases} x^2 \sin(1/x) & x > 0 \\ x^2 \sin(1/x) & x < 0 \end{cases}$$

have a removable discontinuity at zero?
(in other words can I define the function to have a value at $x = 0$ making a continuous function?)

$$n_1(x) = \begin{cases} x^2 \sin(1/x) & x > 0 \\ ? & x = 0 \\ x^2 \sin(1/x) & x < 0 \end{cases}$$

Using continuity to calculate limits.

> **Note** If a function $f(x)$ is continuous on its domain and if a is in the domain of f, then
>
> $$\lim_{x\to a} f(x) = f(a).$$
>
> That is, if a is in the domain of f, we can calculate the limit at a by evaluation.
> If a is not in the domain of f, we can sometimes use the methods discussed in the last lecture to determine if the limit exists or find its value.

Example (a) Find the following limit:

$$\lim_{x \to \frac{\pi}{2}} \frac{x \cos^2 x}{x + \sin x}$$

Composition of functions

Please review Lecture 13 on Composing Functions on the Alegra/Precalculus review page.

We can further expand our catalogue of functions continuous on their domains by considering composition of functions.

Theorem 3 If G is a continuous at a and F is continuous at $G(a)$, then the composite function $F \circ G$ given by $(F \circ G)(x) = F(G(x))$ is continuous at a, and

$$\lim_{x \to a} (F \circ G)(x) = (F \circ G)(a).$$

That is :

$$\lim_{x \to a} F(G(x)) = F(\lim_{x \to a} G(x)).$$

Note that when the above conditions are met, we can calculate the limit by direct substitution.

Recall that the domain of $F \circ G$ is the set of points $\{x \in \text{dom}G | G(x) \in \text{dom}F\}$. Using this and the theorem above we get:

If $f(x) = F(G(x))$, then f is continuous at all points in its domain if G is continuous at all points in its domain and F is continuous at all points in its domain. (Note that we can repeat the process to get the same result for a function of the form $F(G(H(x)))$.)

Example Evaluate the following limit:

$$\lim_{x \to 1} \sqrt{\frac{x^2 + x - 1}{x + 3}}$$

Let $G(x) = \frac{x^2+x-1}{x+3}$ and $F(x) = \sqrt{x}$. Both F and G are continuous on their domains, therefore

$$F(G(x)) = \sqrt{\frac{x^2 + x - 1}{x + 3}}$$

is continuous on its domain. Since $x = 1$ is in the domain, we can calculate the above limit by direct substitution.

$$\lim_{x \to 1} \sqrt{\frac{x^2 + x - 1}{x + 3}} = \sqrt{\frac{1 + 1 - 1}{1 + 3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

Example (a) Find the domain of the following function and determine if it is continuous on its domain?:

$$f(x) = \cos(x^3 + 1).$$

Recall : If $G(x) = x^3 + 1$ and $F(x) = \cos x$, then $F(G(x)) = \cos(x^3 + 1)$. The domain is the $\{x \in \text{Dom.}G | G(x) \in \text{Dom.}F\}$.

(b) What is $\lim_{x \to 5} \cos(x^3 + 1)$?

Intermediate Value Theorem

Here is another interesting and useful property of functions which are continuous on a closed interval $[a, b]$: the function must run through every y-value between $f(a)$ and $f(b)$. This makes sense since, a continuous function can be drawn without lifting the pen from the paper.

Intermediate value Theorem Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ ($f(a) \neq f(b)$), then there exists a number c in the interval (a, b) with $f(c) = N$.

Example If we consider the function $f(x) = x^2 - 1$, on the interval $[0, 2]$, we see that $f(0) = -1 < 0$ and $f(2) = 3 > 0$. Therefore the intermediate value theorem says that there must be some number, c, between 0 and 2 with $f(c) = 0$. The graph of $f(x)$ crosses the x axis at the point where $x = c$. What is the value of c in this case?

$$f(x) = x^2 - 1 = 0, \quad \text{if} \quad x^2 = 1$$

This is true if $x = \pm 1$, therefore for $c = 1 \in [0, 2]$, we have $f(c) = 0$.

In the above case, there is only one such c, however in general c may not be unique. Also it may be difficult to determine the value of c, however the theorem can be used to narrow down where the roots of an equation are.

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\cos x = x^2 \quad (0, \pi)$$

Extra Examples, Please attempt the following problems before looking at the solutions

Example Which of the following functions are continuous on the interval $(0, \infty)$:

$$f(x) = \frac{x^3 + x - 1}{x + 2}, \qquad g(x) = \frac{x^2 + 3}{\cos x}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}, \qquad k(x) = |\sin x|.$$

Example Which of the following functions have a removable discontinuity at $x = 2$?:

$$f(x) = \frac{x^3 + x - 1}{x - 2}, \qquad g(x) = \frac{x^2 - 4}{x - 2}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}.$$

Example Find the domain of the following function and use Theorems 1, 2 and 3 to show that it is continuous on its domain:

$$k(x) = \frac{\sqrt[3]{\cos x}}{x - 10}.$$

Example Evaluate the following limits:

$$\lim_{x \to \pi} \sqrt[3]{2 + \cos x} \qquad \lim_{x \to \frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}}$$

Example What is the domain of the following function and what are the (largest) intervals on which it is continuous?

$$g(x) = \frac{1}{\sqrt{1 - \sqrt{x}}}.$$

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\sqrt[3]{x} = 1 - x \qquad (0, 1).$$

Solutions

Example Which of the following functions are continuous on the interval $(0, \infty)$:

$$f(x) = \frac{x^3 + x - 1}{x + 2}, \qquad g(x) = \frac{x^2 + 3}{\cos x}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}, \qquad k(x) = |\sin x|.$$

Since $f(x)$ is a rational function, it is continuous everywhere except at $x = -2$, Therefore it is continuous on the interval $(0, \infty)$.
By Theorem 2 and the continuity of polynomials and trigonometric functions, $g(x)$ is continuous except where $\cos x = 0$. Since $\cos x = 0$ for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, we have $g(x)$ is not continuous on $(0, \infty)$.
By theorems 2 and 3, $h(x)$ is continuous everywhere except at $x = 2$. In fact $x = 2$ is not in the domain of this function. Hence the function is not continuous on the interval $(0, \infty)$.
Since $k(x) = |\sin x| = F(G(x))$, where $G(x) = \sin x$ and $F(x) = |x|$, we have that $k(x)$ is continuous everywhere on its domain since both F and G are both continuous everywhere on their domains. Its not difficult to see that the domain of k is all real numbers, hence k is continuous everywhere. (What does its graph look like?)

Example Which of the following functions have a removable discontinuity at $x = 2$?:

$$f(x) = \frac{x^3 + x - 1}{x - 2}, \qquad g(x) = \frac{x^2 - 4}{x - 2}, \qquad h(x) = \frac{\sqrt{x^2 + 1}}{x - 2}.$$

$\lim_{x \to 2} f(x)$ does not exist, since $\lim x \to 2(x^3 + x - 1) = 9$ and $\lim x \to 2(x - 2) = 0$. Therefore the discontinuity is not removable.
$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2}(x + 2) = 4$. Therefore the discontinuity at $x = 2$ is removable by defining a piecewise function:

$$g_1(x) = \begin{cases} g(x) & x \neq 2 \\ 4 & x = 2 \end{cases}$$

$\lim_{x \to 2} h(x)$ does not exist, since $\lim_{x \to 2}(\sqrt{x^2 + 1}) = \sqrt{(5)}$ and $\lim x \to 2(x - 2) = 0$. Therefore the discontinuity is not removable.

Example Find the domain of the following function and use Theorems 1, 2 and 3 to show that it is continuous on its domain:

$$k(x) = \frac{\sqrt[3]{\cos x}}{x - 10}.$$

The domain of this function is all values of x except $x = 10$, since $\cos x$ is defined everywhere as is the cubed root function. Theorem 1 says that the cosine function is continuous everywhere and theorem 3 says that $f(x) = \sqrt[3]{\cos x}$ is continuous for all real numbers since the cubed root function is continuous everywhere. Now we see from Theorem 2 that $k(x) = \frac{f(x)}{g(x)}$ is continuous everywhere except where $g(x) = x - 10 = 0$, that is at $x = 10$.

Example Evaluate the following limits:

$$\lim_{x \to \pi} \sqrt[3]{2 + \cos x} \qquad \lim_{x \to \frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}}$$

Since $G(x) = 2 + \cos x$ and $F(x) = \sqrt[3]{x}$ are continuous everywhere, we have $F(Gx))$ is continuous on its domain and we can calculate the first limit by evaluation:

$$\lim_{x \to \pi} \sqrt[3]{2 + \cos x} = \sqrt[3]{2 + \cos \pi} = \sqrt[3]{2 - 1} = 1.$$

As above, we have $\sqrt[3]{\sin x}$ is continuous on its domain, therefore $\lim_{x\to\frac{\pi}{2}} \sqrt[3]{\sin x} = \sqrt[3]{\sin\frac{\pi}{2}} = 1$. Since $\lim_{x\to\frac{\pi}{2}}(x - \frac{\pi}{2}) = 0$, we have $\frac{\sqrt[3]{\sin x}}{x-\frac{\pi}{2}}$ approaches ∞ in absolute value as x approaches $\frac{\pi}{2}$. As $x \to \frac{\pi}{2}^-$, $\sin(x) > 0$, hence $\sqrt[3]{\sin x} > 0$. As $x \to \frac{\pi}{2}^-$, $x - \frac{\pi}{2} < 0$, therefore the quotient has negative values and

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sqrt[3]{\sin x}}{x - \frac{\pi}{2}} = -\infty.$$

Example What is the domain of the following function and what are the (largest) intervals on which it is continuous?

$$g(x) = \frac{1}{\sqrt{1-\sqrt{x}}}.$$

The domain of this function is all x where $\sqrt{1-\sqrt{x}} \neq 0$, i.e. all x where $x \neq 1$. By theorems 3 and 2, the function is continuous everywhere on its domain, therefore it is continuous on the intervals $(-\infty, 1)$ and $(1, \infty)$.

Example use the intermediate value theorem to show that there is a root of the equation in the specified interval:

$$\sqrt[3]{x} = 1 - x \qquad (0, 1).$$

Let $g(x) = \sqrt[3]{x} - 1 + x$. We have $g(0) = -1 < 0$ and $g(1) = 1 > 0$. therefore by the intermediate value theorem, there is some number c with $0 < c < 1$ for which $g(c) = 0$. That is

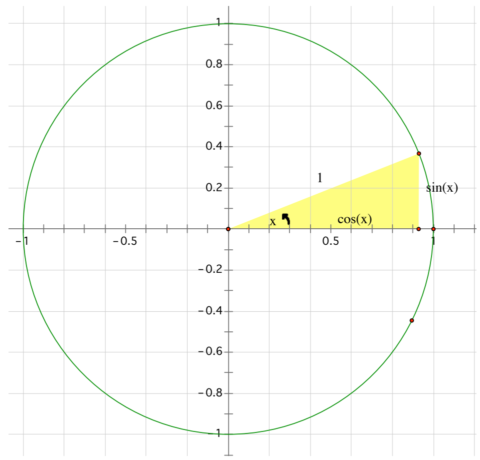
$$\sqrt[3]{c} = 1 - c$$

as desired.

Continuity of Trigonometric Functions

We see the following limits geometrically:

$$\lim_{x\to 0} sin(x) = 0, \qquad \lim_{x\to 0} cos(x) = 1.$$



> **Theorem 2** The functions $\sin x$ and $\cos x$ are continuous on the interval $(-\infty, \infty)$. In particular ; for any real number a, we can evaluate the limits below by direct substitution
>
> $$\lim_{x\to a} \sin x = \sin a, \qquad \lim_{x\to a} \cos x = \cos a.$$

Proof We can use the addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y,$$

to show that $\lim_{x\to a} sin(x) = sin(a)$ and $\lim_{x\to a} cos(x) = sin(a)$ for all real numbers a (radians). We will work through the details in the case of $\sin(x)$.

$$\lim_{x\to a} \sin x = \lim_{h\to 0} \sin(a+h) = \lim_{h\to 0}(\sin a \cos h + \cos a \sin h) = \lim_{h\to 0} \sin a \lim_{h\to 0} \cos h + \lim_{h\to 0} \cos a \lim_{h\to 0} \sin h.$$

$$= (\sin a) \cdot 1 + (\cos a) \cdot 0 = \sin a.$$

Proof of Theorem 2 (5)

> **Theorem 2** If f and g are continuous at a and c is constant, then the following functions are also continuous at a:
>
> **1.** $f+g$ **2.** $f-g$ **3.** cf **4.** fg **5.** $\frac{f}{g}$ if $g(a) \neq 0$.

Proof of 5 : We are assuming that

$$\lim_{x\to a} f(x) = f(a) \quad \text{and that} \quad \lim_{x\to a} g(x) = g(a).$$

Law 5 of our previous lecture on limits says that:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0.$$

Hence $\lim_{x\to a} (f/g)(x) = \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (if $\lim_{x\to a} g(x) = g(a) \neq 0) = \frac{f(a)}{g(a)} = (f/g)(a)$.

Thus $\lim_{x\to a} (f/g)(x) = (f/g)(a)$ and this is what we needed to verify to show that the function f/g is continuous at a. Proofs of properties 1- 4 are easier and similar.
